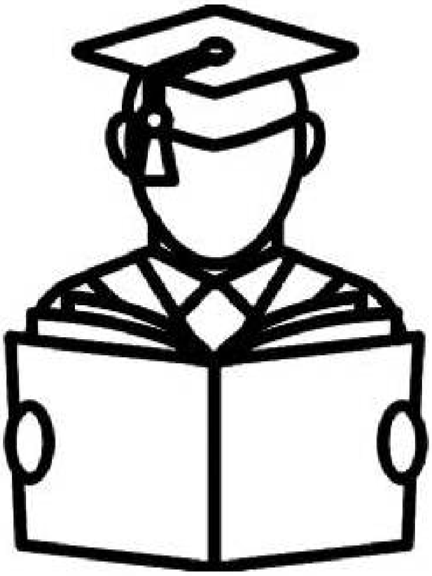
चौधरी PHOTOSTAT

"I don't love studying. I hate studying. I like learning. Learning is beautiful."



"An investment in knowledge pays the best interest."

Hi, My Name is

Mathematical Statistics (MS) for JAM

MATRICES

Definition:- A collection of numbers arranged in rows and columns is said to be an array, A matrix is a rectangular array of numbers closed in addition, substraction, multiplication, and division.

We represent a matrix by $A = [a_{ij}]_{m\times n}$, $i = 1(1)m$, $j = 1(1)n$.

i.e. $\therefore A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{bmatrix}$

Diagonal matrix:-

$A = [a_{ij}]$ is such that $a_{ij} = 0 \; \forall \; i \neq j$

i.e. $A = \text{diag}[d_1, d_2, \ldots, d_n]$

Eg. $A = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{bmatrix}$ is a 3X3 diagonal mtx.

Scalar matrix:- $A = [a_{ij}] \ni a_{ij} = 0 \; \forall \; i \neq j$

$a_{ij} = k \; \forall \; i = j$, $k \in \mathbb{N}$.

Eg $A = \begin{bmatrix} a & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & a \end{bmatrix}$

~~Upper~~ Triangular mtx:- If every element above or below the leading diagonal of a square matrix is zero, then the matrix is called a triangular matrix.

Upper triangular mtx:- $\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}$

Lower Triangular mtx:- $\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$

Equality of matrices:- Two matrices A and B of the same order are said to be equal, if and only if the corresponding elements are equal.

VECTORS AND VECTOR SPACES

FIELD :→ Suppose there is a set F of objects $x, y, z, \ldots$ and two operations on the elements of F as follows. The first operation, called addition, associates with each pair of elements x, y in F an element $(x+y)$, the second operation called multiplication, associates with each pair x, y, an element xy, and these two operations satisfy the following properties :

i) **PROPERTIES OF ADDITION :→**

(a) Closure : $x \in F,\ y \in F \Rightarrow x+y \in F$.

(b) Commutative : $x+y = y+x \Rightarrow x, y \in F$.

(c) Associative : $x+(y+z) = (x+y)+z,\ \forall\ x, y, z \in F$.

(d) Nutral element : There is an unique element zero (0) in F $\ni$
$$x+0 = x\ \forall\ x \in F.$$

(e) Inverse : To each $x \in F$, there corresponds an unique element $(-x)$ in F $\ni$ $x+(-x) = 0$.

ii) **PROPERTIES OF MULTIPLICATION :→**

(a) Closure : $x \in F,\ y \in F, \Rightarrow xy \in F$.

(b) Commutative : $xy = yx \Rightarrow x, y \in F$.

(c) Associative : $x(yz) = (xy)z \Rightarrow x, y, z \in F$

(d) Nutral element : There is an unique non-zero element 1 in F
$$\ni x \times 1 = x\ \forall\ x \in F.$$

(e) Inverse : To each non-zero $x \in F$, there corresponds an unique element x^{-1} (or, $\frac{1}{x}$) in F $\ni$ $xx^{-1} = 1$.

iii) **PROPERTIES OF ADDITION & MULTIPLICATION :→**
(DISTRIBUTIVITY)

Multiplication distributes over addition, i.e.,
$$x \times (y+z) = xy + xz\ \forall\ x, y, z \in F$$

The set F together with these two operations is called a field.

Ex :→ $F = \{0, 1\}$.

LIMIT

Definition of limit :- $\lim_{x \to a} f(x) = l$ if for every $\epsilon > 0$, $\exists$ a 'δ'(>0) such that $|f(x) - l| < \epsilon$ whenever $|x - a| < \delta$.

Theorem:- $\lim_{x \to a} f(x) = l$ iff for every sequence $\{x_n\}$ converges to 'a', i.e. $\lim_{n \to \infty} f(x_n) = l$.

Remark:- Two sequences $\{a_n\}$ and $\{b_n\}$ are said to be equivalent if $\lim_{n \to \infty} \frac{a_n}{b_n} = l \neq 0$.

Similarly, two functions $f(x)$ and $g(x)$ are said to be equivalent if for large x, $\lim_{x \to \infty} \frac{f(x)}{g(x)} = l \neq 0$.

Examples:-

(i) $\frac{\sqrt{n} + n}{n+1} \sim n\sqrt{n}$

(ii) $\sin\frac{1}{n} \sim \frac{1}{n}$.

(iii) $a^{1/n} - 1 \sim \frac{1}{n}$

(iv) $\sqrt{n+1} - \sqrt{n} \sim \frac{1}{\sqrt{n+1} + \sqrt{n}}$

(v) $\sqrt{n+1} - \sqrt{n} \sim \frac{1}{\sqrt{n}}$

(vi) $(n+1)^2 - n^2 \sim n$.

Note:- $\lim_{x \to a} f(x) = l \Leftrightarrow$ for given $\epsilon > 0$, $\exists$ a $\delta > 0 \ni$

$$0 < |x - a| < \delta \Rightarrow |f(x) - l| < \epsilon$$

$$\Leftrightarrow a - \delta < x < a + \delta \Rightarrow l - \epsilon < f(x) < l + \epsilon$$

$$\Leftrightarrow x \in (a - \delta, a + \delta) \Rightarrow f(x) \in (l - \epsilon, l + \epsilon).$$

(except possibly at $x = a$)

In other words, a real number l is a limit of the function f as x approaches to a if for every nbd of l $\exists$ a nbd of 'a' $\ni$ for every x in nbd of 'a', $f(x)$ is in nbd of l.

Ex. If f is given by $f(x) = \begin{cases} \frac{x^2 - a^2}{x^2 + a^2}, & x \neq a \\ 0, & \text{ow} \end{cases}$

then show that $\lim_{x \to a} f(x) = 2a$.

Sol.

$$|f(x) - 2a| < \epsilon$$

$$\Rightarrow \left| \frac{x^2 - a^2}{x - a} - 2a \right| < \epsilon$$

$$\Rightarrow |x - a| < \epsilon$$

Now if we choose a number $\delta \ni 0 \leq \delta \leq \epsilon$, then

$|f(x) - 2a| < \epsilon$ whenever $|x - a| < \delta \Rightarrow \lim_{x \to a} f(x) = 2a$.

• USEFUL FORMULAE ON REAL NUMBERS •

Absolute Value of a Real Number :- $|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$

Thus, we already have $|x| \geq 0$, also by definition
$|x| = |-x|$.

Note :-
1. $|x| = \max(x, -x)$
2. $|-x| = \max(-x, -(-x)) = \max(-x, x) = |x|$
3. $-|x| = \min(x, -x)$
4. $|x|^2 = |-x|^2 = x^2$
5. $|xy| = |x| \cdot |y|$
6. $\left|\frac{x}{y}\right| = \frac{|x|}{|y|}$, provided $y \neq 0$.

Remark:- If a, b are real numbers, then show that $\max(a,b) = \frac{a+b+|a-b|}{2}$ and $\min(a,b) = \frac{a+b-|a-b|}{2}$.

Triangle Inequalities :- For all real numbers x, y show that
(i) $|x+y| \leq |x| + |y|$, and
(ii) $|x-y| \geq ||x| - |y||$.

Proof :- (i) $|x+y|^2 = (x+y)^2 = x^2 + y^2 + 2xy$
$\leq |x|^2 + |y|^2 + 2|x| \cdot |y|$ $[\because xy \leq |xy| = |x| \cdot |y|]$
$= (|x| + |y|)^2$
since the quantities are non-negative, so taking the +ve sq. root,
$|x+y| \leq |x| + |y|$

(ii). Similar method.

Ex. 1. For real nos. $x, a, \epsilon > 0$, show that
(i) $|x| < \epsilon \Leftrightarrow -\epsilon < x < \epsilon$,
(ii) $|x-a| < \epsilon \Leftrightarrow a - \epsilon < x < a + \epsilon$.

Ex. 2. If $a, b \in \mathbb{R}$ be $\ni a < b + \epsilon$ for each $\epsilon > 0$, then $a \leq b$.

Ex. 3. If $a, b \in \mathbb{R}$, show that if $a \leq b + \frac{1}{n}$, $\forall n \in \mathbb{N}$, then $a \leq b$.

Ex. 4. If for any $\epsilon > 0$, $|b-a| < \epsilon$, then $b = a$.

Ex. 5. If $a, b \in \mathbb{R}$ and $a < c$ for each $c > b$, then $a \leq b$.

Sol. Do yourself.

Ex. 6 $[a+b] \geq [a] + [b]$ for all real numbers a, b.

Ex. 7. $[a] + [-a] = \begin{cases} 0, & a \text{ is an integer} \\ -1, & \text{ow} \end{cases}$

• SEQUENCE OF REAL NUMBERS •

The word "Sequence" is used to convey the idea that the things are arranged in order. Before introductiong the concept of sequence in $\mathbb{R}$, we define function or mapping or transformation between two sets A and B.

Let $f: A \to B$ is a mapping or function if for every $x \in A$, there exists ($\exists$) a unique value of $y \in B$. Then the rule f is called a mapping or a function of A into B.
Here we write $y = f(x)$, where $x \in A$ and $y \in B$.
Note that, $y = \pm x$ is not a function, it's a relation.

• **DEFINITION:—** A sequence of real numbers (or, a sequence in $\mathbb{R}$) is a function defined on the set $\mathbb{N}$ of natural numbers, whose range is a subset of the set $\mathbb{R}$ of real numbers; i.e. if for every $n \in \mathbb{N}$ (a set of natural numbers), $\exists$ a real number a_n, then the order set

$$a_1, a_2, \dots ; a_n, \dots$$

is said to define a set of real numbers.

Notation:—
(1) If a_n is the n^{th} term of a sequence, then we write $a_1, a_2, \dots, a_n, \dots$ to describe the sequence.
(2) $f: \mathbb{N} \to \mathbb{R}$ is a sequence.
(3) A sequence f is generally denoted by the symbol $\{f(n)\}$.
e.g. $\{a_n\}$, $\{b_n\}$, $\{x_n\}$, $\{y_n\}$.

Examples:—
(a) If $b \in \mathbb{R}$, the sequence $B = \{b, b, \dots\}$, all of whose terms are equal b, is called the constant sequence $\{b\}$.
(b) Let $f: \mathbb{N} \to \mathbb{R}$ is defined by $f(n) = \frac{n}{n+1}$, $n \in \mathbb{N}$. The sequence is $\left\{\frac{n}{n+1}\right\}$. It is also denoted by $\left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$
(c) Let $f: \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = (-1)^n$, $n \in \mathbb{N}$. The sequence is $\{(-1)^n\}$. It is also denoted by $\{-1, 1, -1, 1, \dots\}$. The range of the sequence is $\{-1, 1\}$.
(d) The celebrated Fibonacci sequence $F = (f_n)$ is given by the inductive definition

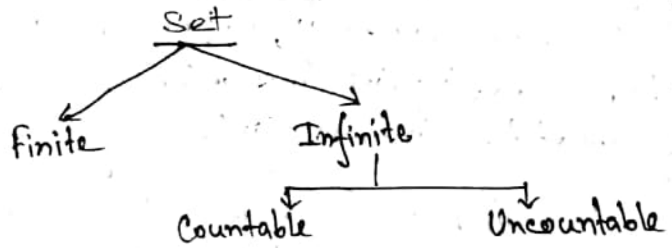
$$f_1 = 1, \; f_2 = 1, \; f_{n+1} = f_{n-1} + f_n \;\; (n \geq 2)$$

Thus each term past the second is the sum of its two immediate predecessors. The sequence is $\{1, 1, 2, 3, 5, 8, 13, 21, 34, \dots\}$.
(e) Null Sequence: A null sequence is one whose terms approach zero, i.e. $\lim a_n = 0 \Rightarrow \{a_n\}$ be a null sequence.
Example:- $\left\{\frac{1}{n}\right\}$ is a null sequence.
Note that, If $\{a_n\}$ be a null sequence then $\{|a_n|\}$ is a null sequence and conversely.

The concept of a SET :> A set is a collection of some elements
which are its members. And the members are called the
elements of the set. Synonyms for set are class, aggregate
and collection. A set can be defined by actually listing its elements
or, if this is not possible, by describing some property held by
all members and by no nonmembers. The first is called the
roster method and the second is called the property method.

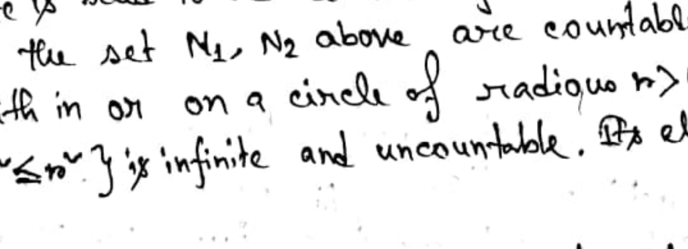
Ex. 1. The set of all vowels in the english alphabet can be defined by
the roster method as {a, e, i, o, u} or by property
method as {x | x is a vowel}, the vertical line | is read
"such that" or "given that".

Ex. 2. The set {x | x is a triangular in a plane} is the set of all
triangles in a plane. Note that the rooster method can't be used
here.



Depending on how many elements it has, a set may be finite
or infinite. The set $M = \{1, 2, \ldots, 50\}$ is finite and contains 50
elements. The set of all natural numbers $N_1 = \{1, 2, \ldots, n, \ldots\}$ is
infinite. The set of all even numbers $N_2 = \{2, 4, \ldots, 2n, \ldots\}$ is also
infinite. An infinite is said to be countable if all its elements can be
enumerated. Both the set N_1, N_2 above are countable. The set
C of all points with in or on a circle of radious $r > 0$,
$C = \{(x, y) : x^2 + y^2 \le r^2\}$ is infinite and uncountable. Its elements can't
be enumerated.

Universal set and Null set :> Empty set is a set containing no
elements; It's a member of all other sets.



The diagram representing a set is called a Venn diagram.

Measure of Central Tendency

C.U

Define Central Tendency? or, What do you mean by Central Tendency of a frequency distribution?

Ans:→ A set of observations shows a tendency or motive to have a value (generally centrally located) by which they may be replaced. This character is termed as Central Tendency.

For any frequency distribution we find a tendency of the variate values to cluster around a central value; in other words, most of the values lies in a small interval about a central value. this characteristic is called the central tendency of a frequency distribution. In relation to a frequency distribution, an average is also termed as a measure of location, because it helps to locate the position of the distribution on the axis of the variable.

Measures of Central Tendency.

Ans:→ Central tendency is measured by —

i) Mean,
ii) Median,
iii) Mode,
iv) Quartile,
v) Decile,
etc.....

Arithmatic Mean.

For Non frequency or raw data.

The arithmatic mean of a variable is derived by dividing the sum of its values by the no. of values. If x denotes the variable under consideration and its values namely $x_1, x_2, \ldots, x_n$ are given, then the arithmatic mean of x, denoted by $\bar{x}$, is given by

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{1}{n}\sum_{i=1}^{n} x_i .$$

For continuous Variable.

Again, for a continuous variable, the data are summarised in a frequency table showing the various class intervals and their corresponding class frequencies. In this case, the class-mark of a class-interval is supposed to represent the interval and on the basis of this assumption, an approximate value of the mean may be obtained. Hence the mean ($\bar{x}$) is expressed in the form

$$\bar{x} = \frac{\sum_{i=1}^{n} x_i f_i}{\sum_{i=1}^{n} f_i} = \frac{\sum_{i=1}^{n} x_i f_i}{N}$$

In the case of equal width of the class intervals, calculation of the mean may be facilitated through a change of origin (or base) and scale. We are to subtract c from each class-mark and then divide the resultant by d, where c is the chosen origin, usually a class-mark near the middle of the range and d, the scale, is the common width. If y_i be the new value corresponding to x_i, then

$$y_i = \frac{x_i - c}{d}$$

or, $x_i = c + d y_i$, for each i

or, $x_i f_i = c f_i + d y_i f_i$, for each i

or, $\sum_i x_i f_i = c \sum_i f_i + d \sum_i y_i f_i$

or, $\frac{1}{n} \sum_i x_i f_i = c + \frac{d}{n} \sum_i y_i f_i$, where $n = \sum_i f_i$

or, $\bar{x} = c + d\bar{y}$.

Calculation of Mean :→

Class Boundaries	Frequency f_i	Class mark x_i	$y_i = \frac{x_i - 55.5}{10}$	$y_i f_i$
30.5–40.5	6	35.5	−2	−12
40.5–50.5	14	45.5	−1	−14
50.5–60.5	20	55.5	0	0
60.5–70.5	7	65.5	1	7
70.5–80.5	3	75.5	2	6
Total =	N=50	—	—	−13

THEORY OF PROBABILITY

Meanings of Probability :- It's a measure of chance of occurance of a phenomenon.

<1> The word 'Probability' may be used to mean 'the degree of belief' of a person making a statement on proposition. It is used in the sense when we say that a certain football team will be the champion in a league or we say that the 'Mahabharat' is very probably the work of several authors.

<2> On the other hand, the word has a different meaning, when we use it in the context of an experiment that can be repeated any no. of times under identical conditions. By the probability of any outcome of the experiment we shall now mean the long run relative frequency of any particular outcome of the experiment. We use the probability in this sense when we say that the probability of getting a 'head' in tossing a coin is 3/4 or the probability that an article produced by a machine will defective is negligable. In statistics, we generally use the term in 2nd sense.

In probability and statistics, we concern ourselves to same special type of experiment.

(1) Random Experiment :—

A random experiment or statistical experiment is an experiment in which—

<i> all possible outcomes of the experiment are known in advance.

<ii> any performance of the experiment results in an outcome that is not known in advance.

<iii> The experiment can be repeated under identical or similar condition.

Ex : Consider an experiment of 'tossing a coin'. If the coin does not stand on the side there are two possible outcomes : Head (H), Tail (T). On any performance of the experiment, one does not know what the result will be. coin can be tossed as many times as desired under identical or similar condition. Hence, tossing of one is a random experiment.

PROBABILITY

"It is a measure of chance of occurrance of a Phenomenon."

NAME:- TANUJIT CHAKRABORTY.

TOPIC:- PROBABILITY THEORY 2.

HONS:- STATISTICS

CLG:- BIDHANNAGAR COLLEGE

YEAR:- B.SC 2ND YEAR

The problem in Probability is —

"Given a stochastic model, what we can say about the outcome".

"STATISTICS"

Spores of Information → $\boxed{\text{Statistical Method}}$ → A valid Decision.

PROBABILITY THEORY 2.

Generating Function :- the generating function of a random variable X, is a function of the form $E(\psi(t,X))$; where t is a non-random variable.

- **P.g.f. (Probability Generating Function) :-** This is meant for a discrete random variable whose mass points are non-negative integers or some subsets of the whole set of non-negative integers. Here $\psi(t,x)$ is of the form t^X. Note that $E(t^X)$ necessarily exists for $|t|=1$. Hence, because of the comparison test we find that the series $p_0 + t.p_1 + t^2 p_2 + \cdots\cdots$ is also absolutely convergent for $|t|<1$. As such the p.g.f. of a non-negative integer valued random variable necessarily exists. It is denoted by $P_X(t)$.

Example:

1. Binomial Distribution :- (with parameter n, p)

$$f(x) = \binom{n}{x} p^x q^{n-x} \quad ; \ x \geq 0$$

$$P(t) = E(t^X)$$
$$= \sum_{x=0}^{n} t^x \binom{n}{x} p^x q^{n-x}$$
$$= \sum_{x=0}^{n} \binom{n}{x} (pt)^x q^{n-x}$$
$$= (q+pt)^n \text{, defined for all real } t.$$

2. Poisson Distribution :- (with parameter λ)

$$f(x) = e^{-\lambda} . \frac{\lambda^x}{x!} \ , x \geq 0 \ ,$$

$$P(t) = E(t^X) = \sum_{x=0}^{\infty} t^x e^{-\lambda} . \frac{\lambda^x}{x!} = \sum_{x=0}^{\infty} e^{-\lambda} \frac{(\lambda t)^x}{x!}$$
$$= e^{-\lambda} \sum_{x} \frac{(\lambda t)^x}{x!}$$
$$= e^{\lambda t} . e^{-\lambda}$$
$$= e^{-\lambda(1-t)} = e^{\lambda(t-1)} .$$

SOME CONTINUOUS DISTRIBUTIONS

Rectangular Distribution OR UNIFORM DISTRIBUTION :—

An absolutely continuous random variable X defined over $[a,b]$, $-\infty < a < b < \infty$ is said to follow uniform distribution with parameter a, b; if its pdf is given by,

$$f(x) = \begin{cases} \frac{1}{b-a}, & a \le x \le b, \\ 0, & \text{ow} \end{cases}$$

We will write $X \sim U[a,b]$ if X has a uniform distribution on $[a,b]$

This distribution is also called a rectangular distribution since the area under 'f' in between a and b is rectangular. It is also called Rectangular Distribution.

$$X \sim U[a,b]$$
or,
$$X \sim R[a,b]$$

The end point a or b or both may be excluded. Clearly,

$$\int_{-\infty}^{\infty} f(x)\,dx = 1.$$

Distribution Function: The DF of X is given by,—

$$F(x) = \begin{cases} 0 & \text{if } x \le a \\ \frac{x-a}{b-a} & \text{if } a < x < b \\ 1 & \text{if } x \geq b \end{cases};$$

Expectation & variance:—

$$E(X^k) = \int_a^b x^k f(x)\,dx = \frac{1}{b-a}\int_a^b x^k\,dx \quad , k>0 \text{ is an integer.}$$

$$= \frac{1}{b-a}\left[x^{k+1}\right]_a^b / k+1$$

$$= \frac{b^{k+1} - a^{k+1}}{(k+1)(b-a)}.$$

Putting $k=1$, $E(X) = \frac{b+a}{2}$,

Putting $k=2$, $E(X^2) = \frac{(b^2+ab+a^2)}{3}$

$\therefore$ $Var(X) = E(X^2) - E^2(X) = \frac{(b-a)^2}{12}.$

NOTE:- The fact can be used to draw random observations from the theoritical distribution of X.

Here at first we choose 3 digited random numbers and put a decimal point before the first digit. Let us denote such a quantity by p, clearly p is a realization from $R(0,1)$ distn., now to obtain x we equate $F(x)=p$ and solve for x.

Theorem: 2. Let F be any DF, and let X be a $U[0,1]$ RV. Then there exists a function h such that $h(X)$ has DF F, i.e.,

$$P\{h(X)\le x\}=F(x), \quad \text{for all } x \in (-\infty,\infty).$$

Proof:- If F is the DF of a discrete RV Y, let

$$P[Y=y_k]=p_k, \quad k=1,2,\ldots$$

Define h as follows:-

$$h(x)=\begin{cases} y_1 & \text{if } 0\le x<p_1, \\ y_2 & \text{if } p_1\le x<p_1+p_2, \\ \vdots & \vdots \end{cases}$$

Then

$$P\{h(X)=y_1\}=P\{0\le X<p_1\}=p_1,$$
$$P\{h(X)=y_2\}=P\{p_1\le X<p_1+p_2\}=p_2,$$

and in general,

$$P\{h(X)=y_k\}=p_k, \quad k=1,2,\ldots.$$

Thus $h(X)$ is a discrete RV with DF F.

If F is continuous and strictly increasing, F^{-1} is well defined, and if we take $h(X)=F^{-1}(X)$. We have

$$P\{h(X)\le x\}=P\{F^{-1}(X)\le x\}$$
$$=P\{X\le F(x)\}$$
$$=F(x),$$

as asserted.

In general, define

$$F^{-1}(y)=\inf\{x: F(x)\ge y\},$$

and let $h(X)=F^{-1}(X)$. Then we have

$$\{F^{-1}(y)\le x\}=\{y\le F(x)\}.$$

$F^{-1}(y)\le x \Rightarrow \forall\ \epsilon>0,\ y\le F(x+\epsilon)$, since $\epsilon>0$ is arbitrary and F is continuous on the right, we let $\epsilon\to 0$ and conclude that $y\le F(x)$. Since $y\le F(x) \Rightarrow F^{-1}(y)\le x$. Thus,

$$P[F^{-1}(X)\le x]=P[X\le F(x)]=F(x).$$

NOTE:- It is quite useful theorem in generating samples with the help of the uniform distribution.

SOME PROBABILITY INEQUALITIES

The inequalities which contain probability in either left side or right side or in the both side, are called "Probability Inequalities".

MARKOV'S INEQUALITY :—

Statement :— Let X be a r.v. having finite expectation, i.e. $E(X)$ converges. then for any non-zero quantity 'a', we have the inequality :

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

Proof :— Let us define a r.v. Y such that

$$Y = \begin{cases} a & \text{if } X \geq a \\ 0 & \text{ow} \end{cases}$$

$$X \geq Y \Rightarrow E(X) \geq E(Y)$$

Now, $E(Y) = a \cdot P(X \geq a) \leq E(X)$

$$\Rightarrow P(X \geq a) \leq \frac{E(X)}{a}.$$

NOTE: Markov inequality holds for any function of r.v. X, i.e. for any real valued function $g(X)$, the markov's inequality is given by

$$P[g(X) \geq a] \leq \frac{E(g(X))}{a}, \quad a \neq 0$$

Proof :— Let us define a function of r.v. Y, $g(Y)$

$$g(Y) = \begin{cases} a & \text{if } g(X) \geq a \\ 0 & \text{ow} \end{cases}$$

$$g(X) \geq g(Y)$$

$$\therefore E(g(X)) \geq E(g(Y))$$

$$\therefore E(g(Y)) = a \cdot P[g(X) \geq a] \leq E(g(X))$$

$$\Rightarrow P[g(X) \geq a] \leq \frac{E(g(X))}{a}, \quad a \neq 0$$

Problem 1. If X be any r.v. such that $M(t)=E(e^{tX})$ exists for all t, show that for any $t>0$,

$$P(tX > s^2+\ln M(t)) < e^{-s^2}$$

ANS:- We know that an exponential function is monotonically increasing.

So, $P(tX > s^2+\ln M(t))$

$$= P(e^{tX} > e^{s^2+\ln M(t)})$$

$$= P[e^{tX} > e^{s^2}.e^{\ln M(t)}]$$

Let $g(X)=e^{tX}$ then by Markov's inequality, we have

$$P(e^{tX} > e^{s^2}.e^{\ln M(t)}) < \frac{E(e^{tX})}{e^{s^2}.e^{\ln M(t)}} = \frac{M(t)}{M(t).e^{s^2}}$$

$$= e^{-s^2}$$ (Proved)

Problem 2. For any random variable X, show that,

$$P[|X|>t] \le \frac{1+t^2}{t^2} E\left(\frac{X^2}{1+X^2}\right) \text{ for any } t>0.$$

[2002]

ANS:- Here, $P[|X|>t]$

$$= P[X^2 > t^2]$$

$$= P[1+X^2 > 1+t^2]$$

$$= P\left[\frac{X^2}{1+X^2} > \frac{t^2}{1+t^2}\right]$$

Now by Markov's inequality,

$$P\left[\frac{X^2}{1+X^2} > \frac{t^2}{1+t^2}\right] \le E\left(\frac{X^2}{1+X^2}\right).\frac{1+t^2}{t^2}$$ [Proved]

C.U. S.T. $P[X>t] < E(e^{aX})/e^{at}$.

ANS:- $P[aX>at] = P[X>t]$

$= P(e^{aX} > e^{at})$ [$\because$ e is monotonically increasing]

By Markov's inequality,

$< \frac{E(e^{aX})}{e^{at}}$, where $E(e^{aX})$ exists where $a>0$.

Problem 3. A fair die is rolled n times. Find a lower bound to n such that, the probability of at least one six in rolling is $\geq \frac{1}{2}$.

ANS:- Let us define a random variable X representing the numbers of six by throwing a die n times.

$\therefore X \sim bin(n, \frac{1}{6})$

By Markov's inequality,

$$P[X \geq 1] \leq \frac{E(X)}{1}$$

$$\Rightarrow P[X \geq 1] \leq \frac{n}{6} \text{ ——————— } \langle i \rangle$$

Again it is given that $P[X \geq 1] \geq \frac{1}{2}$

$\therefore$ From $\langle i \rangle$,

$$\frac{n}{6} \geq \frac{1}{2}$$

$$\Rightarrow n \geq 3$$

$\therefore$ the die should be at least thrown 3 times.

Problem 4. $X_1, X_2, \ldots, X_K$ are independent r.v.'s having zero mean and unit variance. Find an upper bound to,

$$P\left[\sum_{i=1}^{K} X_i^2 \geq \lambda K\right], \ \lambda > 0$$

ANS:- $X_1, X_2, \ldots, X_K$ are independent r.v.'s with mean 0 and variance 1.

i.e. $E(X_i) = 0 \ \forall \ i = 1(1)K$.

$$V(X_i) = E(X_i^2) - E^2(X_i)$$

$$\Rightarrow E(X_i^2) = 1 \qquad [\because E(X_i) = 0]$$

$$\Rightarrow \sum_{i=1}^{K} E(X_i^2) = K$$

$$\Rightarrow E\left(\sum_{i=1}^{K} X_i^2\right) = K \qquad [\because X_i\text{'s are independent}]$$

Now by Markov's inequality,

$$P\left[\sum_{i=1}^{K} X_i^2 \geq \lambda K\right] \leq \frac{E\left(\sum_{i=1}^{K} X_i^2\right)}{\lambda K} = \frac{K}{\lambda K} = \frac{1}{\lambda}$$

$\therefore$ Required upper bound $= \frac{1}{\lambda}$.

CHEBYSHEV'S INEQUALITY : —

Statement :– For a random variable X having finite mean and variance σ^2, then for any $t > 0$, the Chebyshev's inequality is given as follows:

$$P(|X-\mu| \geq t\sigma) \leq \frac{1}{t^2}$$

or

$$P(|X-\mu| \leq t\sigma) \geq 1 - \frac{1}{t^2}$$

Proof :— In order to prove Chebyshev's inequality, we will first prove Markov's inequality,

Let us define a random variable Z,

$$Z = \begin{cases} a, & Y \geq a \\ 0, & \text{ow} \end{cases}$$

where Y is another RV.

From the definition of Z, it is such that,

$$Y \geq Z$$

$$\Rightarrow E(Y) \geq E(Z)$$

$$\Rightarrow E(Y) \geq a \cdot P[Y \geq a]$$

$$\Rightarrow P[Y \geq a] \leq \frac{E(Y)}{a}$$

This is the required Markov's inequality.

Now for the RV X,

$E(X) = \mu < \infty$, $V(X) = \sigma^2 = E(X-\mu)^2 > 0$

Now, $P[|X-\mu| \geq t\sigma] = P[(X-\mu)^2 \geq t^2\sigma^2]$

Now, let us choose $Y = (X-\mu)^2$ and $a = t^2\sigma^2$, then by Markov's inequality, we have,

$$P[(X-\mu)^2 \geq t^2\sigma^2] \leq \frac{E(X-\mu)^2}{t^2\sigma^2} = \frac{\sigma^2}{t^2\sigma^2}$$

$$= \frac{1}{t^2}$$

$$\therefore P[|X-\mu| \geq t\sigma] \leq \frac{1}{t^2} \quad \text{——} \langle i \rangle$$

Hence proved.

1 – ⟨i⟩ gives

$$P[|X-\mu| \leq t\sigma] \geq 1 - \frac{1}{t^2} \quad \text{——} \langle ii \rangle$$

Hence proved.

REAL/MATHEMATICAL ANALYSIS

- SEQUENCES OF REAL NUMBERS: The word "sequence" is used to convey the idea that the things are arranged in order.

Definition:→ A 'sequence' of real numbers is a function defined on the set N of natural numbers whose range is a subset of the set $\mathbb{R}$ of real numbers; i.e. if for every $n \in N$, $\exists$ a real number a_n, then the order set

$$a_1, a_2, \dots, a_n, \dots$$

is said to define a sequence of real nos.

Remark:→ 1) $f: A \to B$ is a mapping or function if for every $x \in A$, $\exists$ a unique value of $y \in B$.

Here, we write $y = f(x)$ where $x \in A$, $y \in B$.

2) $y = \pm x$ is not a function, it's a relation.

Notation:→ 1) If a_n is the n^{th} term of a sequence, then we write $a_1, a_2, \dots, a_n, \dots$, to describe the sequence.

2) $f: N \to \mathbb{R}$ is a sequence.

3) $\{a_n\}, \{b_n\}, \{x_n\}, \{y_n\}$.

The main question we are concerned with here is to decide whether or not the term a_n tends to a finite quantity when n increases indefinitely.

Definition:→ A sequence $\{a_n\}$ is said to have a limit $l \in \mathbb{R}$ if, for every $\epsilon > 0$, $\exists$ a natural number $N(\epsilon)$, $\ni |a_n - l| < \epsilon$, for all $n \geq N(\epsilon)$

Example.(1) Prove that $\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0$.

Soln.→ Let $\epsilon > 0$ be an arbitrary number, then

$$|a_n - l| < \epsilon$$

$$\Rightarrow \left|\frac{1}{n} - 0\right| < \epsilon$$

$$\Rightarrow \frac{1}{n} < \epsilon$$

$$\Rightarrow n > \frac{1}{\epsilon}$$

In particular if $\epsilon = 0.1$, then

$$\left|\frac{1}{n} - 0\right| < \epsilon = 0.1$$

$$\Rightarrow n > \frac{1}{\epsilon} = 10$$

$$\therefore \left|\frac{1}{n} - 0\right| < \epsilon = 0.1$$

whenever $n \geq 11 = N(\epsilon = 0.1)$

If $\epsilon = 0.01$, then

$$\left|\frac{1}{n} - 0\right| < \epsilon = 0.01$$

$$\Rightarrow n > \frac{1}{\epsilon} = 100$$

$$\Rightarrow n > 101$$

Hence take $N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$

- Or, choose a natural no. $N(\epsilon)$ which is $> \frac{1}{\epsilon}$.

Then $\exists$ a natural no. $N(\epsilon) \ni \left|\frac{1}{n} - 0\right| < \epsilon \;\; \forall \; n \geq N(\epsilon)$

Hence, by definition, $\lim\limits_{n\to\infty} \left(\frac{1}{n}\right) = 0$

$$|a_n - l| < \epsilon \text{ for } n \geq N(\epsilon)$$

$$\Rightarrow l - \epsilon < a_n < l + \epsilon \text{ for } n \geq N(\epsilon)$$

For $0 < \epsilon' < \epsilon$, then in general,

$$N(\epsilon') > N(\epsilon)$$

If ϵ is small, then $N(\epsilon)$ will be sufficiently large to ensure $|a_n - l| < \epsilon$ for $n \geq N(\epsilon)$. And all the members a_n, $n \geq N(\epsilon)$ are in the small interval, $(l - \epsilon, l + \epsilon)$ i.e. then a_n is very close to l.

If a sequence $\{a_n\}$ has a finite limit 'l', then we say that the sequence $\{a_n\}$ converges to l or the sequence is convergent. If a sequence does not converge to a finite limit, then it is said to be divergent. If $\{a_n\}$ converges to 'l'; we write

$$\lim_{n\to\infty} (a_n) = l, \text{ or, } \lim (a_n) = l.$$

<u>Example. (2).</u> Prove that $\lim\limits_{n\to\infty} \left(\frac{1}{n+1}\right) = 0$.

<u>Soln.</u> $\rightarrow$ Let $\epsilon > 0$ be an arbitrary number,

Then $|a_n - l| < \epsilon$

$$\Rightarrow \left|\frac{1}{n+1} - 0\right| < \epsilon$$

$$\Rightarrow \frac{1}{n+1} < \epsilon$$

$$\Rightarrow n < \frac{1}{\epsilon} - 1$$

Choose a natural number $N(\epsilon)$ which is $> \frac{1}{\epsilon} - 1$.

Then, $\exists$ a natural no. $N(\epsilon) \ni \left|\frac{1}{n+1} - 0\right| < \epsilon$ for all $n \geq N(\epsilon)$

By defn. $\lim\limits_{n\to\infty} \left(\frac{1}{n+1}\right) = 0$

<u>Alt.</u> $\rightarrow$ $\left|\frac{1}{n+1} - 0\right| = \frac{1}{n+1} < \frac{1}{n} < \epsilon$

$$\Rightarrow n > \frac{1}{\epsilon}$$

Take $N(\epsilon) = \left[\frac{1}{\epsilon}\right] + 1$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni \left|\frac{1}{n+1} - 0\right| < \epsilon \;\; \forall \; n \geq N(\epsilon)$

$\therefore$ By defn. $\lim\limits_{n\to\infty} \left(\frac{1}{n+1}\right) = 0$.

Example (3) Prove that $\lim_{n\to\infty}\left(\frac{1}{n^2+1}\right)=0$

Soln. → Let $\epsilon>0$ be an arbitrary number.

$$|a_n-l|<\epsilon$$

$$\Rightarrow \left|\frac{1}{n^2+1}-0\right| = \frac{1}{n^2+1} < \frac{1}{n^2} < \epsilon$$

$$\Rightarrow n \geq \frac{1}{\sqrt{\epsilon}}$$

Take $N(\epsilon) = \left[\frac{1}{\sqrt{\epsilon}}\right]+1$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni$ $\left|\frac{1}{n^2+1}-0\right| < \epsilon \quad \forall\ n \geq N(\epsilon)$.

$\therefore$ By defn. $\lim_{n\to\infty}\left(\frac{1}{n^2+1}\right)=0$.

Example (4). Prove that $\lim_{n\to\infty}\left(\frac{2n^2+1}{n^2+n}\right)=2$.

Soln. → Let $\epsilon>0$ be an arbitrary number.

$$\therefore |a_n-l|<\epsilon$$

$$\Rightarrow \left|\frac{2n^2+1}{n^2+n}-2\right| < \epsilon$$

$$\Rightarrow \frac{\cancel{2n^2}+1-\cancel{2n^2}-2n}{n^2+n} < \epsilon$$

$$\Rightarrow \frac{1-2n}{n^2+n} < \epsilon$$

$$\Rightarrow \frac{2n-1}{n^2+n} < \frac{2n}{n^2} = \frac{2}{n} < \epsilon$$

$$\Rightarrow n > \frac{2}{\epsilon}.$$

Take $N(\epsilon) = \left[\frac{2}{\epsilon}\right]+1$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni$ $\left|\frac{2n^2+1}{n^2+n}-2\right| < \epsilon \ \forall\ n \geq N(\epsilon)$

$\therefore$ By defn. $\lim_{n\to\infty}\left(\frac{2n^2+1}{n^2+n}\right)=2$.

Example (5). Prove that $\lim_{n\to\infty}\left(\frac{1}{n^p}\right)=0$, $p>0$.

Soln. → Let $\epsilon>0$ be an arbitrary number.

$$|a_n-l|<\epsilon$$

$$\Rightarrow \left|\frac{1}{n^p}-0\right|<\epsilon$$

$$\Rightarrow \frac{1}{n^p} < \epsilon$$

$$\Rightarrow n^p > \frac{1}{\epsilon}$$

$$\Rightarrow n > \left(\frac{1}{\epsilon}\right)^{1/p}$$

since $p>0$, Take $N(\epsilon) = \left[\frac{1}{\sqrt[p]{\epsilon}}\right]+1$.

Then $\exists$ a natural no. $N(\epsilon)$, $\ni$ $\left|\frac{1}{n^p}-0\right| < \epsilon \ \forall\ n \geq N(\epsilon)$

Example (7). Prove that $\lim_{n\to\infty}(r^n)=0$ if $|r|<1$.

Soln.→ Let $\epsilon>0$ be an arbitrary number.

Then If $|r^n-0|<\epsilon$

$\Rightarrow |r^n|<\epsilon$

$\Rightarrow n\ln|r| < \ln\epsilon$

$\Rightarrow n > \frac{\ln\epsilon}{\ln|r|}$ [since $|r|<1$, $\ln|r|<0$]

Choose a natural no. $N(\epsilon)$ which is $\geq \frac{\ln\epsilon}{\ln|r|}$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni |r^n-0|<\epsilon$ $\forall$ $n\geq N(\epsilon)$

$\therefore$ By defn. $\lim_{n\to\infty}(r^n)=0$ if $|r|<1$.

Example (8). Prove that $\lim_{n\to\infty}\left(2-\frac{1}{2^n}\right)=2$

Soln.→ Let $\epsilon>0$ be an arbitrary no.

$\therefore \left|2-\frac{1}{2^n}-2\right|<\epsilon$

$\Rightarrow \left|-\frac{1}{2^n}\right|<\epsilon$

$\Rightarrow \frac{1}{2^n}<\frac{1}{n}<\epsilon$

$\Rightarrow n>\frac{1}{\epsilon}$

Take $N(\epsilon)=\left[\frac{1}{\epsilon}\right]+1$

Then $\exists$ a natural no. $N(\epsilon)$, $\ni \left|2-\frac{1}{2^n}-2\right|<\epsilon$ $\forall$ $n\geq N(\epsilon)$

$\therefore$ By defn. $\lim_{n\to\infty}\left(2-\frac{1}{2^n}\right)=2$.

Example (9). Prove that $\lim_{n\to\infty}(2^{1/n})=1$.

Soln.→ Let $\epsilon>0$ be an arbitrary no.

$\therefore |2^{1/n}-1|<\epsilon$

$\Rightarrow 2^{1/n}<\epsilon+1$

$\Rightarrow \frac{1}{n}\ln 2<\ln(\epsilon+1)$

$\Rightarrow \frac{1}{n}<\frac{\ln(\epsilon+1)}{\ln 2}$

$\Rightarrow n>\frac{\ln 2}{\ln(\epsilon+1)}$

Take $N(\epsilon)=\left[\frac{\ln 2}{\ln(\epsilon+1)}\right]$

$\therefore n>N(\epsilon)$. [P]

RANDOM SAMPLING AND SAMPLING DISTRIBUTION

Definition of Some Terms :

1. Parameter : A constant which changes its value from one situation to another. Specially, it is denoted by θ. A parameter labels a distribution uniquely.

2. Parameter Space : Set of all admissible values of the parameter, denoted by Θ

Example : i) $X \sim N(\mu, 1)$
μ = Parameter,
$\mathbb{R}$ = Parameter space.

ii) $X \sim N(\mu, \sigma^2)$
(μ, σ) = Parameter (vector valued)
Parameter Space = $\{(\mu, \sigma) : \mu \in \mathbb{R}, \sigma \in \mathbb{R}^+\}$

iii) $X \sim Bin(n, p)$
(n, p) = Parameter
Parameter space = $\{(n, p) : n \in \mathbb{N}, 0 < p < 1\}$

3. Labelling Parameter : Suppose X is normally distributed with mean μ and s.d. unity. Then the parameter μ labels the distribution uniquely and hence termed as labelling parameter.

On the other hand, the parameter $\xi_{1/2}$, the median of a distribution though reflects a feature (regarding location) of the distribution, but it fails to label the distribution. But in case of one parameter Cauchy distribution with median θ, which labels the distribution.

Thus if a random variable X has distribution Function F, where the distribution is labelled or, indexed by the parameter θ. We denote the distn. by $F_\theta(\cdot)$.

Family of Distribution :

Let X be a random variable having distribution function F_θ, $\theta \in \Theta$, then $\{F_\theta(\cdot) : \theta \in \Theta\}$ is said to be a family of distribution function, similarly, one may define a family of PDF or PMF's namely $\{f_\theta : \theta \in \Theta\}$, where $f_\theta(\cdot)$ is the PDF or PMF of X.

Example : $\{\Phi(x-\mu) : \mu \in \mathbb{R}\}$
is a family of normal distribution with mean μ and s.d. unity.

4. <u>Random Sample</u> : – If $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variable each having distribution function F then $(X_1, X_2, \ldots, X_n)$ constitutes a random sample drawn from F.

5. <u>Sample Space</u> : – Let $(X_1, X_2, \ldots, X_n)$ be a random sample drawn from a distribution having distribution function F. Suppose $(x_1, x_2, \ldots, x_n)$ is the realization on $(X_1, X_2, \ldots, X_n)$ then $(x_1, x_2, \ldots, x_n)$ is said to be a sample point. Clearly, these sample points may vary from one sampling location to another. The totality of all such sample points constitutes the sample space, commonly denoted by $\mathfrak{X}$.

<u>Example</u>: Suppose, we have a random sample of size 2 from $N(\mu, 1)$ distribution. Then the sample space will be $\mathbb{R}^2$.

6. <u>Statistic</u> : – Let $(X_1, X_2, \ldots, X_n)$ be a random sample drawn from a population having distribution function $F_\Theta(\cdot)$. suppose $T(X_1, X_2, \ldots, X_n)$ is a measurable function $\ni$ $T : \mathbb{R}^n \rightarrow \mathbb{R}^k$ if $k=1$, T is said to be a real valued statistic and for $k>1$, T will be a vector valued statistic.

In simple words, statistic is a function of sample observation which is independent of any unknown parameters, i.e. here T does not depend on the labelling parameter Θ.

<u>Example</u>: Let $(X_1, X_2, \ldots, X_n)$ be a random sample drawn from $N(\mu, 1)$ population.

Here, the sample mean $\bar{X}$ is a statistic, we know, $\bar{X} \sim N(\mu, \frac{1}{n}) \Rightarrow \sqrt{n}(\bar{X} - \mu) \sim N(0,1)$.

It is to be noted that unless μ is specified, $\sqrt{n}(\bar{X} - \mu)$ would not be a statistic.

Once μ is specified as 2, $\sqrt{n}(\bar{X} - 2)$ becomes a statistic.

Some real valued statistic are sample mean $\bar{X}$, sample Range $\mathbb{R}$, sample s.d. 'S'.

$X_{(1)}$, the minimum of the sample observation,

$X_{(n)}$, the maximum of the sample observation,

where as $(\bar{X}, S)$, $(X_{(1)}, X_{(n)})$, $(X_{(1)}, X_{(2)}, \ldots, X_{(n)})$ are vector valued statistic.

[7]. Sampling Distribution: → The probability distribution of any statistic is termed as sampling distribution.

(2006) In a problem of parametric inference the population feature of interest can objectively be written as a function of labelling parameter, say $\gamma(\theta)$, where, an observation $X_1 \sim F_\theta$, which is not known completely except the form of F.

Now in order to guess $\gamma(\theta)$ [Problems of estimation] or, to validate any conjuctured regarding $\gamma(\theta)$ [Problem of hypothesis testing], we proceed with a specific statistic and make use of its sampling distribution. In such inferential problem we always associate a measure of error with the conclusion where this error is nothing but the sampling error. As a measure the s.d. of the sampling distribution of the statistic would serve the purpose and this is termed as the standard error.

Exhibiting a sampling distribution in case of sampling from a finite identifiable population: —

Here the term identifiability means that the population units can easily be distinguished.

[8]. Simple Random Sampling: → Suppose we have a finite identifiable population of size N $(U_1, U_2, \ldots, U_N)$, where U_α is the αth member of the population. By a sample we mean, a non-empty collection of units from $(U_1, U_2, \ldots, U_N)$ with, or, without repetations. Here the sampling procedure may be subjective (purposive sampling, deliberate sampling, haphazard sampling) or, objective (Probabilistic, non-probabilistic, mixed).

The probabilistic sampling may be an equal probability sampling where each of the possible sample has the same probability to occur (or, every unit of the population has the same probability to be included in the sample) or, an unequal probability sampling.

- Definition: → Simple random sampling (SRS) is an equal probability sampling. An SRS may be drawn with replacement termed as SRSWR or, without replacement termed as SRSWOR.

SRSWR : — Suppose a sample of n units is drawn from the population of size N one by one with replacement. Clearly, number of possible samples is N^n on each has probability $\frac{1}{N^n}$ to occur.

SRSWOR : — Suppose a sample of n units is drawn at random one by one without replacement from the population of size N. If we ignore the order of the units in the sample. The number of possible sample will be $\binom{N}{n}$ and each has probability $\frac{1}{\binom{N}{n}}$ to occur.

On the other hand, if the order is taken into account the number of possible sample is $(N)_n$ and each has the probability $\frac{1}{(N)_n}$ to occur.

Suppose an SRSWOR of size 3 is drawn from a population of size 5.

$(u_1, u_2, u_3, u_4, u_5)$

Let us ignore the order of the units in the sample.

Further assume that the variate values of the population units are 6, 8, 4, 6, 8, respectively.

Let s be a typical sample and $\bar{x}(s)$ be the sample mean. Then we have the following sampling distribution of the sample mean.

Serial No.	s	$\bar{x}(s)$
1	(u_1, u_2, u_3)	6
2	(u_1, u_2, u_4)	6.67
3	(u_1, u_2, u_5)	7.33
4	(u_1, u_3, u_4)	5.33
5	(u_1, u_3, u_5)	6
6	(u_1, u_4, u_5)	6.67
7	(u_2, u_3, u_4)	6
8	(u_2, u_3, u_5)	6.67
9	(u_2, u_4, u_5)	7.33
10	(u_3, u_4, u_5)	6

$\bar{x}$	
5.33	1/10
6	2/5
6.67	3/10
7.33	1/5